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MOMENTS OF POWER TRANSFORMED TIME SERIES

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SUMMARY

A simple recursion is presented for calculating moments (e.g., mean, variance, and autocorrelation function) of a time series that has been power transformed to normality. Its derivation is elementary, relying on the moment-generating function for a bivariate normal distribution. To make clear the distinction between the moments of the transformed and original time series, the special case of a squared normal process is treated in detail. The use of the recursion is illustrated through an environmental example, motivated by stochastic modeling of hourly precipitation, that involves both stabilization of variance and amplification of autocorrelation. Copyright © 1999 John Wiley & Sons, Ltd.

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1. INTRODUCTION

In the analysis of environmental data, the application of a transformation to normality is a popular technique for dealing with non-normal distributions (Stoline 1991). This approach is also used for other purposes, such as stabilizing the variance. It is especially convenient for dealing with environmental time series that are autocorrelated. Nevertheless, it is often desired to obtain statistical properties of the original, untransformed series (Shumway *et al*. 1989). Even if the issue of transformation bias (Neyman and Scott 1960) is ignored, determining these properties can be tedious in practice. Moreover, much of the theory for time series that are transformed to normality relies on Hermite polynomial expansions (Granger and Newbold 1976), a technical topic not ordinarily treated in introductory statistics courses and with which environmental scientists will not ordinarily be familiar.

The present paper presents a simple recursion for the moments (e.g., mean, variance, and autocorrelation function) of a time series whose power transformation is normally distributed. This recursion is limited to the situation in which the power transformation is the inverse of an integer, including the square, cube, and fourth root that are commonly applied to environmental time series. Because simple analytical expressions exist for the moments of a lognormal time series (Granger and Newbold 1976), this particular transformation is not treated here.

It is explained in Section 2 that only the moments of a bivariate normal distribution are needed to determine the moments of a time series whose power transformation is normally distributed. By making use of the moment-generating function for the bivariate normal distribution, a recursion for these moments is derived in Section 3. To make clear the distinction between the moments of the transformed and original time series, the special case of a squared normal process

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is treated in detail in Section 4. The use of the recursion is illustrated in Section 5 through an environmental example, motivated by stochastic modeling of hourly precipitation (Katz and Parlange 1995). Finally, Section 6 consists of a brief discussion.

2. POWER TRANSFORMED TIME SERIES

Let the time series $\{Y_t: t = 1, 2, \ldots\}$ be related to a normally distributed series $\{X_t: t = 1, 2, \ldots\}$ by

$$X_t = Y_t^s, \ 0 < s < 1, \ \text{or} \ Y_t = X_t^r, \ 0 < r < \infty, \tag{1}$$

a form of the Box–Cox transformation (Box and Cox 1964). In (1), $r = 1/s$ is assumed to be a positive integer (i.e., corresponding to such popular values of s as 1/2, 1/3, or 1/4). Experience in analyzing environmental time series indicates that the condition of s being positive is not overly restrictive. Further, limiting r to integral values is consistent with the principle of adopting a transformation that is simple to interpret. Note that, if (1) is reexpressed in the equivalent form of $(Y_t^s - 1)/s$, then the limiting case of $s = 0$ corresponds to the logarithmic transformation mentioned earlier. For clarity, Y_t is referred to as the 'original' series, whereas X_t is the 'transformed' series (but note that Granger and Newbold (1976) employed the reverse terminology).

The transformed time series has mean, variance, and autocorrelation function denoted by $\mu_X(t) = E(X_t)$, $\sigma_X^2(t) = \text{var}(X_t)$, and $\rho_X(l;\ t) = \text{corr}(X_t,\ X_{t+l})$, $l = 1,\ 2, \ldots,$ respectively, all possibly depending on the index of time t. Given a model that specifies these moments of the transformed series, it is desired to obtain the corresponding moments for the original series, denoted by $\mu_Y(t)$, $\sigma_Y^2(t)$, and $\rho_Y(l;\ t)$, respectively. Because

$$E(Y_t) = E(X_t^r), \ E(Y_t^2) = E(X_t^{2r}), \ \text{and} \ E(Y_t Y_{t+l}) = E(X_t^r X_{t+l}^r), \tag{2}$$

certain moments of univariate and bivariate normal distributions are needed. For the square root transformation (i.e., $s = 1/2$), the expressions for the moments of the original series are reasonably straightforward to derive and to interpret (see Section 4). Unfortunately, the brute force derivation of the higher-order moments of a bivariate normal distribution required for smaller values of s (i.e., larger values of r) is tedious, and the resultant expressions for the moments of the original series are not necessarily very enlightening.

It is important to recognize that no simple correspondence exists between the mean, variance, or autocorrelation function of the original and transformed time series. In particular, the mean and variance of the original series are each nonlinear functions of *both* the mean and variance of the transformed series. Likewise, the autocorrelation function of the original series depends not only on the autocorrelation function, but on the means and variances as well, of the transformed series. Examples of these relationships are provided in Sections 4 and 5.

3. RECURSION FOR MOMENTS

The moment-generating function of a bivariate normal distribution (say, the joint distribution of two random variables Z_i with means μ_i, variances σ_i^2, $i = 1, 2$, and correlation coefficient ρ) can be expressed as

$$M(u_1, u_2) = \exp[\mu_1 u_1 + \mu_2 u_2 + (1/2)(\sigma_1^2 u_1^2 + 2\sigma_1 \sigma_2 \rho u_1 u_2 + \sigma_2^2 u_2^2)]. \tag{3}$$

The so-called product-moments for this distribution are denoted by

$$\gamma(j,k) = E(Z_1^j Z_2^k), \; j,k = 0,1,\ldots \; . \tag{4}$$

Repeated partial differentiation of (3) and evaluation of the resultant expressions at $u_1 = u_2 = 0$ yields the following two-dimensional recursion, listed in an order convenient for programming:

$$\begin{aligned}
\gamma(j,0) &= \mu_1\gamma(j-1,0) + (j-1)\sigma_1^2\gamma(j-2,0),\\
\gamma(0,k) &= \mu_2\gamma(0,k-1) + (k-1)\sigma_2^2\gamma(0,k-2),\\
\gamma(1,k) &= \mu_2\gamma(1,k-1) + \sigma_1\sigma_2\rho\gamma(0,k-1) + (k-1)\sigma_2^2\gamma(1,k-2),\\
\gamma(j,k) &= \mu_1\gamma(j-1,k) + k\sigma_1\sigma_2\rho\gamma(j-1,k-1) + (j-1)\sigma_1^2\gamma(j-2,k),
\end{aligned} \tag{5}$$

j, k = 2, 3,.... From the definition of moments, the initial conditions for (5) are $\gamma(0, 0) = 1$, $\gamma(1, 0) = \mu_1$, $\gamma(0, 1) = \mu_2$, and $\gamma(1, 1) = \mu_1\mu_2 + \sigma_1\sigma_2\rho$. Note that the first two equations in (5) provide the univariate moments of Z_i, $i = 1, 2$. (A BASIC program of this recursion is available from the author upon request; e-mail: rwk@ucar.edu.)

To obtain the autocorrelation coefficient at lag l of the original time series, $\rho_Y(l; t)$, the recursion (5) would be utilized with the following parameters for the bivariate normal distribution:

$$\mu_1 = \mu_X(t), \; \mu_2 = \mu_X(t+l), \; \sigma_1^2 = \sigma_X^2(t), \; \sigma_2^2 = \sigma_X^2(t+l), \; \rho = \rho_X(l;t). \tag{6}$$

In view of (2), the recursion would need to be iterated until the following moments are produced: $\gamma(r, 0)$, $\gamma(0, r)$, $\gamma(2r, 0)$, $\gamma(0, 2r)$, and $\gamma(r, r)$.

It should be noted that, in the special case of zero means and unit variances, a recursion for the moments of a bivariate normal distribution can be found in Stuart and Ord (1987, p. 114). But it would be somewhat tedious to convert these central moments to the desired noncentral moments.

4. SQUARED NORMAL TIME SERIES

In the case of a power transformation value of $s = 1/2$, it is feasible to derive expressions for the moments of the original time series directly. Alternatively, they can be obtained from the recursion presented in Section 3. This 'squared normal' series has been commonly employed for modeling environmental time series; for example, in modeling wind speeds (Brown *et al.* 1984; Haslett and Raftery 1989). Granger and Newbold (1976) have given expressions for the moments of a squared normal series that is stationary.

The mean and variance of the original series, using expressions for $\gamma(2, 0)$, $\gamma(0, 2)$, $\gamma(4, 0)$, and $\gamma(0, 4)$ from the recursion (5), are given by

$$\mu_Y(t) = \mu_X^2(t) + \sigma_X^2(t), \; \sigma_Y^2(t) = 2\sigma_X^2(t)[2\mu_X^2(t) + \sigma_X^2(t)]. \tag{7}$$

Of course, the expression for the mean of the original series is just the well-known relationship between the central and noncentral second moments of the transformed series.

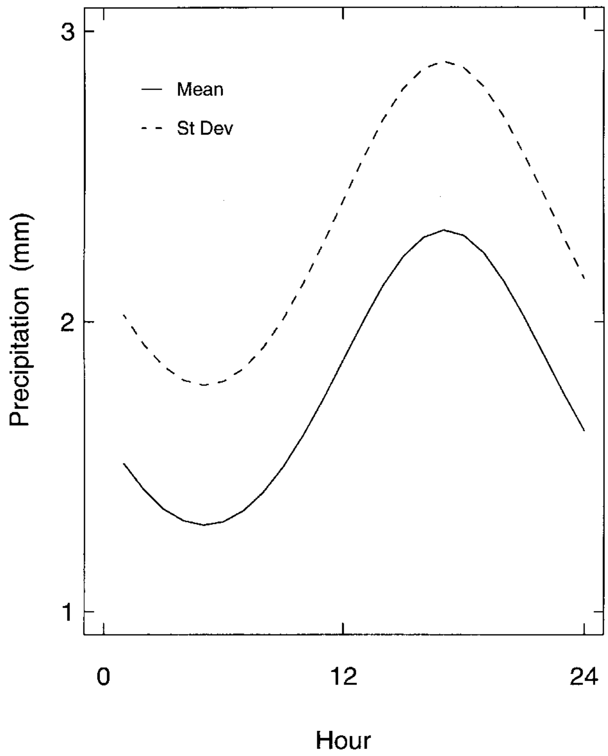

Figure 1. Hourly mean and standard deviation of original time series for precipitation example

Using an expression for $\gamma(2, 2)$ from the recursion (5), the autocorrelation function of the original series is given by

$$\rho_Y(l; t) = w_1 \rho_X(l; t) + w_2 \rho_X^2(l; t). \tag{8}$$

Here

$$\begin{aligned} w_1 &= [4\mu_X(t)\mu_X(t+l)\sigma_X(t)\sigma_X(t+l)]/[\sigma_Y(t)\sigma_Y(t+l)], \\ w_2 &= [2\sigma_X^2(t)\sigma_X^2(t+l)]/[\sigma_Y(t)\sigma_Y(t+l)], \end{aligned} \tag{9}$$

and $\sigma_Y(t)$ can be determined from (7). The more complex structure of the autocorrelation function of the original series is evident, being a weighted sum of the autocorrelation function of the transformed series and its square. These weights (9) depend on both the mean and variance of the transformed series, and satisfy $|w_1 + w_2| \leqslant 1$. Consequently, the autocorrelation function of the original series can be no larger (in absolute value) than that for the original series. Nevertheless, if the means of the transformed series are much larger than the corresponding variances, then it follows from (8) and (9) that the autocorrelation function of the original series would resemble that for the transformed series. These results have been noted by Granger and Newbold (1976) for the case in which the squared normal series is stationary.

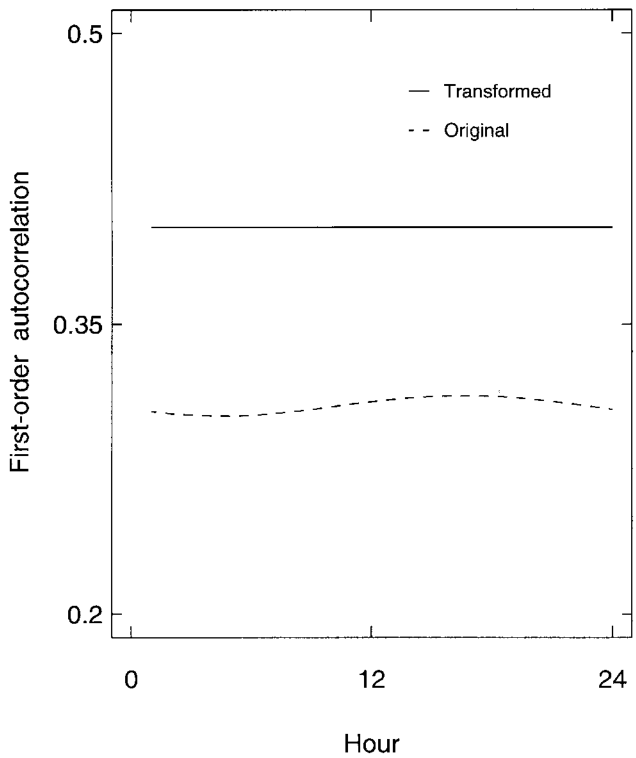

Figure 2. Hourly first-order autocorrelation coefficient for original and transformed time series for precipitation example

5. PRECIPITATION EXAMPLE

Katz and Parlange (1995) utilized power transformations in formulating a stochastic model for hourly precipitation. Conditional on occurrence, the hourly precipitation amount Y_t (termed 'intensity') is power transformed (via (1)) to allow for the relatively high degree of positive skewness of its distribution. To permit a diurnal cycle in the distribution of transformed hourly intensity, its mean is modeled as a cosine wave, which can be expressed as

$$\mu_X(t) = A + B\cos(2\pi t/24) + C\sin(2\pi t/24), \quad t = 1, 2, \ldots . \tag{10}$$

It is further assumed that the variance of transformed intensity $\sigma_X^2(t) = \sigma_X^2$, independent of time t (although Katz and Parlange considered the possibility that it has a diurnal cycle as well). Finally, the deviations about the cyclic mean (10) are modeled as a first-order autoregression (AR(1)) process; that is, with autocorrelation function $\rho_X(l;\ t) = \rho_X^l(1),\ l = 1,\ 2, \ldots,$ also independent of t.

The parameter values for the transformed series (based on $s = 1/8$) were selected to resemble the estimates obtained by Katz and Parlange: $A = 1\,\mathrm{mm}^{1/8}$, $B = -0{\cdot}01\,\mathrm{mm}^{1/8}$, $C = -0{\cdot}04\,\mathrm{mm}^{1/8}$, $\sigma_X = 0{\cdot}15\,\mathrm{mm}^{1/8}$, and $\rho_X(1) = 0{\cdot}4$. For such a value of s, analytical expressions for the moments would be extremely complex. Nevertheless, the recursion (5) can be applied to calculate the numerical values of the moments. The mean $\mu_Y(t)$ and standard deviation $\sigma_Y(t)$ for the original series are shown in Figure 1. Despite the lack of any diurnal cycle in the standard deviation of the

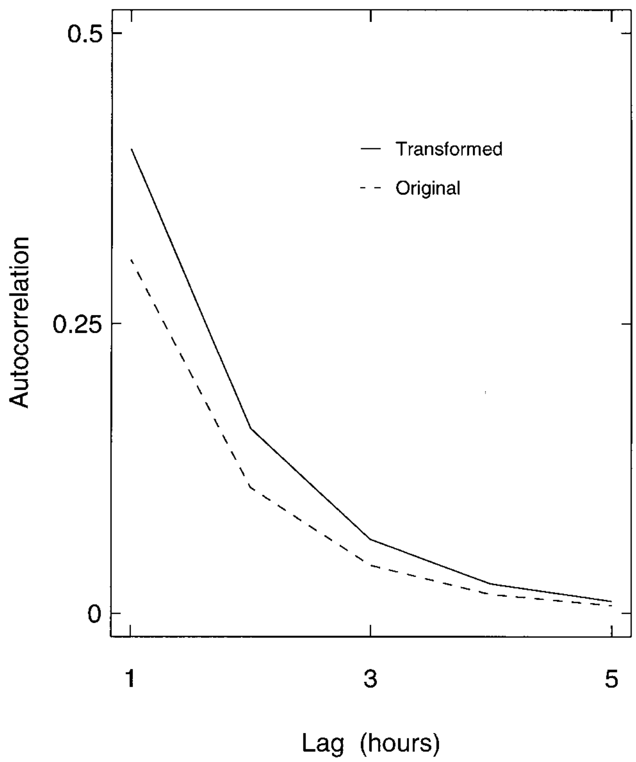

Figure 3. Autocorrelation functions of original and transformed time series for precipitation example ($t = 1$)

transformed series, a marked diurnal cycle is present in the standard deviation for the original series. This cycle is induced by the diurnal cycle in the mean of the transformed series alone, illustrating how the power transformation serves to stabilize the variance.

As depicted in Figure 2, the first-order autocorrelation coefficient $\rho_Y(1; t)$ for the original series (calculated via (5)) is about 0·30 or 0·31, strictly smaller than that for the transformed series, and with virtually no diurnal cycle. Figure 3 shows the autocorrelation function of the original series when $t = 1$ (i.e., $\rho_Y(l; 1)$, $l = 1, 2, \ldots$). This function is smaller than the corresponding one for the transformed series at all lags. Moreover, it decreases toward zero at a rate that is slightly slower than the geometric one for the transformed series (an almost imperceptible difference in Figure 3). This effect of amplified autocorrelations (or 'enhanced predictability') has already been noted for a nonstationary squared normal process (Section 4), and has been shown to hold for a wide class of transformations to stationary normal series (Granger and Newbold 1976).

6. DISCUSSION

The lognormal transformation is routinely applied to many environmental time series to remove positive skewness. Nevertheless, the resultant distribution is sometimes not the normal and not even symmetric, but negatively skewed. The application of a power transformation, with a power between zero and one, is a natural alternative in this situation. In this paper, a simple recursion is presented for calculating the moments of such time series. Its use is illustrated through an example concerned with the stochastic modeling of hourly precipitation. This example requires the treatment of diurnal cycles, whereas other environmental applications would deal with seasonal cycles or trends.

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